ELECTRICAL SIMULATION OF NONSTATIONARY HEAT TRANSFER IN A DISPERSE SYSTEM

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The system of differential equations of heat conduction of a dispersed medium is replaced by the system of differential equations of an electrical circuit; the simulation conditions are examined.

In [1, 3]* it was shown that the differential equations of heat conduction in a disperse system can be written in the following form:

$$\frac{\partial t}{\partial \tau} = a' \frac{\partial^2 t}{\partial x^2} + \beta' (\theta - t), \tag{1}$$

$$\frac{\partial \theta}{\partial \tau} = a'' \frac{\partial^2 \theta}{\partial x^2} - \beta'' (\theta - t). \tag{2}$$

Solutions of system (1), (2) were also obtained in the form of sums of infinite series.

An analysis of the solutions of the system of equations (1), (2) showed that at small values of time τ the corresponding series converged extremely slowly. On the other hand, an analysis of the system is of greatest interest precisely at small values of τ, when the heat fluxes are considerable and there is least justification for assuming that the dispersed medium may be regarded as a continuous medium with appropriate values of the thermophysical coefficients. The solutions obtained can be analyzed numerically on a digital computer. However, initially the complete analysis of the solution requires the comparison of numerous variants with variation of the individual parameters in order to estimate the possibility of simplifications and compare the solutions obtained with the experimental data. In this case the use of a digital computer would involve a great deal of time.

Accordingly we selected the less laborious and sufficiently reliable method of electrical simulation, which, moreover, permits a discrete system to be more correctly simulated if a suitable step is chosen.

We will consider the electrical circuit shown in Fig. 1. Here, two conducting bars are separated by an intermediate layer F and insulated from the walls by a gap l. We assume that the capacitance between the bars can be neglected as compared with the conductance in the intermediate layer F and that the conductance between the bars and the wall can be neglected as compared with the capacitance.

Let A' and A'' be the conductances of the corresponding bars referred to unit length. Then in this case the conductances of an element of the bars of length Δx are, respectively, equal to $A'/\Delta x$ and $A''/\Delta x$. Assume that the functions $A'(x)$ and $A''(x)$ are continuous and differentiable with respect to the independent variable x. We will now consider certain points O' and O'' on the two bars and the adjacent points $1'$ and $2'$ and $1''$ and $2''$ at distances from the points O' and O'' equal to $+\Delta x$ and $-\Delta x$, respectively. We denote the conductance between the bars per unit length by B and assume that it is independent of x. Then on the length Δx the conductance between the bars is equal to $B\Delta x$, and the capacitance to $C'\Delta x$ and $C''\Delta x$, respectively. We will consider the equivalent circuit of the elementary cell Δx (Fig. 2).

Following Gutenmakher [2], we derive the differential equation of voltage distribution along bars of variable conductance. From Kirchhoff's law (Fig. 2) we have the equations

$$i_1' - i_2' = i_4 - i_3,$$
$$i_1'' - i_2'' = i_5 + i_3.$$

We determine the corresponding currents as follows:

$$i_1' = \frac{A_1'}{\Delta x}(U_1 - U_0),$$

$$i_2' = \frac{A_2'}{\Delta x}(U_0 - U_2),$$

$$i_1'' = \frac{A_1''}{\Delta x}(V_1 - V_0),$$

$$i_2'' = \frac{A_2''}{\Delta x}(V_0 - V_2),$$

$$i_3 = B\Delta x(V_0 - U_0),$$

$$i_4 = C'\Delta x \frac{\partial U_0}{\partial \tau},$$

$$i_5 = C''\Delta x \frac{\partial V_0}{\partial \tau}.$$

Then the current increment, divided by Δx, will be

$$\frac{\Delta i'}{\Delta x} = \frac{A_1'}{\Delta x}\frac{U_1 - U_0}{\Delta x} - \frac{A_2'}{\Delta x}\frac{U_0 - U_2}{\Delta x} =$$

$$= \frac{1}{(\Delta x)^2}\left[A_1'(U_1 - U_0) + A_2'(U_2 - U_0)\right] =$$

$$= C' \frac{\partial U_0}{\partial \tau} - B(V_0 - U_0), \tag{3}$$

$$\frac{\Delta i''}{\Delta x} = \frac{A_1''}{\Delta x}\frac{V_1 - V_0}{\Delta x} - \frac{A_2''}{\Delta x}\frac{V_0 - V_2}{\Delta x} =$$

*After publication of these articles the authors received from Professor L. I. Rubinshtein a copy of his paper [4] from which it follows that the proposed system of differential equations was obtained by another method in 1945.

$$= \frac{1}{(\Delta x)^2}\left[A_1''(V_1 - V_0) + A_2''(V_2 - V_0)\right] =$$

$$= C''\frac{\partial V_0}{\partial \tau} + B(V_0 - U_0). \quad (4)$$

Denoting $U_k - U_0$ by ΔU_k and $V_k - V_0$ by ΔV_k, we write Eqs. (3) and (4) in the abbreviated form

$$\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k'\Delta U_k = D', \quad (5)$$

$$\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k''\Delta V_k = D'', \quad (6)$$

where

$$D' = C'\frac{\partial U_0}{\partial \tau} - B(V_0 - U_0),$$

$$D'' = C''\frac{\partial V_0}{\partial \tau} + B(V_0 - U_0).$$

Applying the Taylor expansion formula to the differences ΔU_k and ΔV_k, we see that the first differences ΔU_1 and ΔU_2, and also ΔV_1 and ΔV_2, if we stop at the fourth-order term in the expansion, have the following form:

$$\frac{\Delta U_2}{\Delta x} = \frac{\partial U}{\partial x} + \frac{\Delta x}{2!}\frac{\partial^2 U}{\partial x^2} + \frac{(\Delta x)^2}{3!}\frac{\partial^3 U}{\partial x^3} + \frac{(\Delta x)^3}{4!}\left(\frac{\partial^4 U}{\partial x^4}\right)_{2,0}, \quad (7)$$

$$\frac{\Delta U_1}{\Delta x} = -\frac{\partial U}{\partial x} + \frac{\Delta x}{2!}\frac{\partial^2 U}{\partial x^2} - \frac{(\Delta x)^2}{3!}\frac{\partial^3 U}{\partial x^3} + \frac{(\Delta x)^3}{4!}\left(\frac{\partial^4 U}{\partial x^4}\right)_{1,0}, \quad (8)$$

$$\frac{\Delta V_2}{\Delta x} = \frac{\partial V}{\partial x} + \frac{\Delta x}{2!}\frac{\partial^2 V}{\partial x^2} + \frac{(\Delta x)^2}{3!}\frac{\partial^3 V}{\partial x^3} + \frac{(\Delta x)^3}{4!}\left(\frac{\partial^4 V}{\partial x^4}\right)_{2,0}, \quad (9)$$

$$\frac{\Delta V_1}{\Delta x} = -\frac{\partial V}{\partial x} + \frac{\Delta x}{2!}\frac{\partial^2 V}{\partial x^2} - \frac{(\Delta x)^2}{3!}\frac{\partial^3 V}{\partial x^3} + \frac{(\Delta x)^3}{4!}\left(\frac{\partial^4 V}{\partial x^4}\right)_{1,0}, \quad (10)$$

where $(\partial^4 U/\partial x^4)_{k,l}$, $(\partial^4 V/\partial x^4)_{k,l}$ denote the mean value of the fourth derivative for the interval x_k, x_l.

Substituting (7) and (8), as well as (9) and (10), in (5) and (6), respectively, we obtain

$$\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k'\Delta U_k = \frac{A_2' - A_1'}{\Delta x}\frac{\partial U}{\partial x} + \frac{A_2' + A_1'}{2!}\frac{\partial^2 U}{\partial x^2} + \frac{A_2' - A_1'}{3!}\Delta x\frac{\partial^3 U}{\partial x^3} + \frac{A_2' + A_1'}{4!}(\Delta x)^2\left[\left(\frac{\partial^4 U}{\partial x^4}\right)_{2,0} + \left(\frac{\partial^4 U}{\partial x^4}\right)_{1,0}\right], \quad (11)$$

$$\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k''\Delta V_k = \frac{A_2'' - A_1''}{\Delta x}\frac{\partial V}{\partial x} + \frac{A_2'' + A_1''}{2!}\frac{\partial^2 V}{\partial x^2} + \frac{A_2'' - A_1''}{3!}\Delta x\frac{\partial^3 V}{\partial x^3} + \frac{A_2'' + A_1''}{4!}(\Delta x)^2\left[\left(\frac{\partial^4 V}{\partial x^4}\right)_{2,0} + \left(\frac{\partial^4 V}{\partial x^4}\right)_{1,0}\right]. \quad (12)$$

Now applying the Taylor formula to the difference $(A_2' - A_1')$ and also to $(A_2'' - A_1'')$, discarding terms of the third order and above, and introducing the notation $(A_2' + A_1')/2 = A'$ and $(A_2'' + A_1'')/2 = A''$ we obtain

$$\frac{A_2' - A_1'}{\Delta x} = \frac{dA'}{dx} + \frac{\Delta x}{2!}\frac{d^2 A'}{dx^2}, \quad (13)$$

$$\frac{A_2'' - A_1''}{\Delta x} = \frac{dA''}{dx} + \frac{\Delta x}{2!}\frac{d^2 A''}{dx^2}. \quad (14)$$

Substituting the differences (13) and (14) in (11) and (12), we obtain

$$\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k'\Delta U_k = \frac{dA'}{dx}\frac{\partial U}{\partial x} + A'\frac{\partial^2 U}{\partial x^2} + \frac{\Delta x}{2!}\frac{d^2 A'}{dx^2}\frac{\partial U}{\partial x} + \frac{(\Delta x)^2}{3!}\frac{dA'}{dx}\frac{\partial^3 U}{\partial x^3} + \frac{(\Delta x)^3}{2!3!}\frac{\partial^3 U}{\partial x^3}\left(\frac{\partial^3 U}{\partial x^3}\right)_{1,2} + A'\frac{2(\Delta x)^2}{4!}\left[\left(\frac{\partial^4 U}{\partial x^4}\right)_{2,0} + \left(\frac{\partial^4 U}{\partial x^4}\right)_{1,0}\right], \quad (15)$$

$$\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k''\Delta V_k = \frac{dA''}{dx}\frac{\partial V}{\partial x} + A''\frac{\partial^2 V}{\partial x^2} + \frac{\Delta x}{2!}\frac{d^2 A''}{dx^2}\frac{\partial V}{\partial x} + \frac{(\Delta x)^2}{3!}\frac{dA''}{dx}\frac{\partial^3 V}{\partial x^3} + \frac{(\Delta x)^3}{2!3!}\frac{\partial^3 V}{\partial x^3}\left(\frac{\partial^3 V}{\partial x^3}\right)_{1,2} + A''\frac{2(\Delta x)^2}{4!}\left[\left(\frac{\partial^4 V}{\partial x^4}\right)_{2,0} + \left(\frac{\partial^4 V}{\partial x^4}\right)_{1,0}\right]. \quad (16)$$

In the limit as $\Delta x \to 0$, we obtain

$$\lim_{\Delta x \to 0}\left[\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k'\Delta U_k\right] = \frac{dA'}{dx}\frac{\partial U}{\partial x} + A'\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(A'\frac{\partial U}{\partial x}\right); \quad (17)$$

similarly,

$$\lim_{\Delta x \to 0}\left[\frac{1}{(\Delta x)^2}\sum_{k=1}^{2} A_k''\Delta V_k\right] = \frac{dA''}{dx}\frac{\partial V}{\partial x} + A''\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(A''\frac{\partial V}{\partial x}\right). \quad (18)$$

Assuming in this particular case that A' and A'' are constants and substituting the values of D' and D'' in (5) and (6), we finally obtain

$$C'\frac{\partial U}{\partial \tau}=A'\frac{\partial^2 U}{\partial x^2}+B(V-U), \qquad (19)$$

$$C''\frac{\partial V}{\partial \tau}=A''\frac{\partial^2 V}{\partial x^2}-B(V-U). \qquad (20)$$

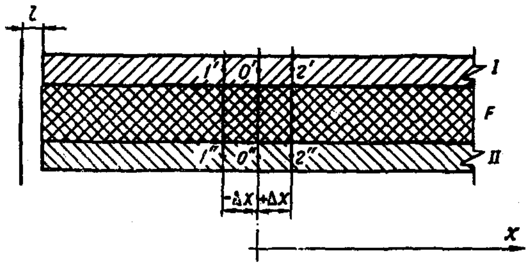

Fig. 1. Electrical analog of heat conduction in a dispersed medium.

Dividing Eqs. (19) and (20) by C' and C", respectively, and introducing the notation $A'/C' = k'$, $A''/C'' = k''$, $B/C' = p'$ and $B/C'' = p''$, we obtain

$$\frac{\partial U}{\partial \tau}=k'\frac{\partial^2 U}{\partial x^2}+p'(V-U), \qquad (21)$$

$$\frac{\partial V}{\partial \tau}=k''\frac{\partial^2 V}{\partial x^2}-p''(V-U). \qquad (22)$$

In order to find the similarity conditions, we reduce Eqs. (1) and (2), together with (21) and (22), to dimensionless form, expressing the variables in the form of a product of the initial (basic) quantities and dimensionless multipliers. Thus, for example, for Eqs. (1) and (2) of the thermal system we have

$$a'=\bar{a}'a'_0,\ \beta'=\bar{\beta}'\beta'_0,\ t=\bar{t}\,t_0,\ \theta=\bar{\theta}\theta_0,$$
$$a''=\bar{a}''a''_0,\ \beta''=\bar{\beta}''\beta''_0,\ \tau=\bar{\tau}\tau_{0T},\ x=\bar{x}x_0.$$

Here, β'_0, t_0, and the other quantities with subscript 0 are basic values, and $\bar{\beta}'$, $\bar{t}$, and the other barred quantities dimensionless multipliers. Substituting these products in Eqs. (1) and (2) and placing the basic quantities in front of the differentiation sign, we obtain

$$\left[\frac{t_0}{\tau_{0T}}\right]\frac{\partial \bar{t}}{\partial \tau_T}=$$
$$=\left[a'_0\frac{t_0}{x_0^2}\right]\bar{a}'\frac{\partial^2 \bar{t}}{\partial \bar{x}^2}+[\beta'_0\theta_0]\,\bar{\beta}'\bar{\theta}-[\beta'_0 t_0]\,\bar{\beta}'\bar{t}, \qquad (23)$$

$$\left[\frac{\theta_0}{\tau_{0T}}\right]\frac{\partial \bar{\theta}}{\partial \tau_T}=$$
$$=\left[a''_0\frac{\theta_0}{x_0^2}\right]\bar{a}''\frac{\partial^2 \bar{\theta}}{\partial \bar{x}^2}+[\beta''_0 t_0]\,\bar{\beta}''\bar{t}-[\beta''_0\theta_0]\,\bar{\beta}''\bar{\theta}. \qquad (24)$$

Dividing the terms of Eqs. (23) and (24) by t_0/τ_{0T} and θ_0/τ_{0T}, respectively, we see that all the terms of the equations assume a dimensionless form, namely:

$$\frac{\partial \bar{t}}{\partial \bar{\tau}_T}=\left[a'_0\frac{\tau_{0T}}{x_0^2}\right]\bar{a}'\frac{\partial^2 \bar{t}}{\partial \bar{x}^2}+$$
$$+\left[\frac{\beta'_0\theta_0\tau_{0T}}{t_0}\right]\bar{\beta}'\bar{\theta}-[\beta'_0\tau_{0T}]\,\bar{\beta}'\bar{t}, \qquad (25)$$

$$\frac{\partial \bar{\theta}}{\partial \bar{\tau}_T}=\left[a''_0\frac{\tau_{0T}}{x_0^2}\right]\bar{a}''\frac{\partial^2 \theta}{\partial \bar{x}^2}+$$
$$+\left[\frac{\beta''_0 t_0\tau_{0T}}{\theta_0}\right]\bar{\beta}''\bar{t}-[\beta''_0\tau_{0T}]\,\bar{\beta}''\bar{\theta}. \qquad (26)$$

After making analogous calculations for the electrical system, we obtain the following system of dimensionless equations:

$$\frac{\partial \bar{U}}{\partial \tau_E}=\left[k'_0\frac{\tau_{0E}}{x_0^2}\right]\bar{k}'\frac{\partial^2 \bar{U}}{\partial x^2}+$$
$$+\left[\frac{p'_0V_0\tau_{0E}}{U_0}\right]\bar{p}'\bar{V}-[p'_0\tau_{0E}]\,\bar{p}'\bar{U}, \qquad (27)$$

$$\frac{\partial \bar{V}}{\partial \tau_E}=\left[k''_0\frac{\tau_{0E}}{x_0^2}\right]\bar{k}''\frac{\partial^2 \bar{V}}{\partial x^2}+$$
$$+\left[\frac{p''_0U_0\tau_{0E}}{V_0}\right]\bar{p}''\bar{U}-[p''_0\tau_{0E}]\,\bar{p}''\bar{V}. \qquad (28)$$

Consequently, the circuit in question does in fact simulate the system of differential equations (1) and (2). For similarity of the phenomena in the thermal and electrical systems, which are expressed by the same differential equations, it is necessary for all the coefficients of these equations, reduced to dimensionless form, to be respectively equal. Then the changes of the unknown quantity in the electrical circuit $[\bar{U}(x,\tau_E)$ and $\bar{V}(x,\tau_E)]$, referred to its basic value, will coincide with the changes of the analogous quantity of the thermal system $[\bar{t}(x,\tau_T)$ and $\bar{\theta}(x,\tau_T)]$, referred to its basic value. Thus, equality of the following quantities is required:

$$a'_0\frac{\tau_{0T}}{x_0^2}=k'_0\frac{\tau_{0E}}{x_0^2}, \qquad (29)$$

$$\frac{\beta'_0\theta_0\tau_{0T}}{t_0}=\frac{p'_0V_0\tau_{0E}}{U_0}, \qquad (30)$$

$$\beta'_0\tau_{0T}=p'_0\tau_{0E}, \qquad (31)$$

$$a''_0\frac{\tau_{0T}}{x_0^2}=k''_0\frac{\tau_{0E}}{x_0^2}, \qquad (32)$$

$$\frac{\beta''_0t_0\tau_{0T}}{\theta_0}=\frac{p''_0U_0\tau_{0E}}{V_0}, \qquad (33)$$

$$\beta''_0\tau_{0T}=p''_0\tau_{0E}. \qquad (34)$$

From (31) and (34) we obtain

$$\beta'_0/\beta''_0=p'_0/p''_0=C''_0/C'_0. \qquad (35)$$

From (29), (32) we obtain

$$\frac{A'_0}{A''_0}=\frac{a'_0}{a''_0}\,\frac{C'_0}{C''_0}.$$

Using (35), we obtain

$$\frac{A_0'}{A_0''} = \frac{a_0'}{a_0''}\,\frac{\beta_0''}{\beta_0'}. \qquad (36)$$

In practice relations a'_0/a''_0 and β''_0/β'_0 are known for each specific case. Thus, we have a relation between A'_0 and A''_0. Selecting one of these quantities, we determine the other, together with C'_0 and C''_0, thus:

$$C_0' = \frac{A_0'}{a_0'}\,\frac{\tau_{0E}}{\tau_{0T}}, \qquad (37)$$

$$C_0'' = \frac{A_0''}{a_0''}\,\frac{\tau_{0E}}{\tau_{0T}}. \qquad (38)$$

Knowing the values of C'_0 and C''_0, we find value of B_0 from (31) or (34). From (31) it follows that

$$B_0 = \beta_0' C_0' \frac{\tau_{0T}}{\tau_{0E}}. \qquad (39)$$

Thus, having selected a suitable ratio of the durations of the natural and simulated processes, we can calculate all the quantities required for the electrical circuit (Fig. 2).

NOTATION

a' is the thermal diffusivity of the solid phase; a'' is the thermal diffusivity of the gas phase; θ is the temperature of the gas; t is the temperature of the particles; τ is the time; C' and C" denote electrical capacitance; U is the potential at bar I; V is the potential at bar II; A' and A" are the conductances of bars referred to unit length; B is the conductance of intermediate layer F per unit length.

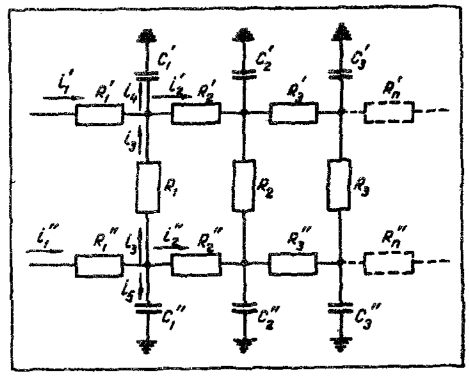

Fig. 2. Electrical model of thermal system.

REFERENCES

1. N. V. Antonishin, L. E. Simchenko, and G. A. Surkov, in: Investigation of Heat and Mass Transfer in Technical Processes and Equipment [in Russian], izd. Nauka i tekhnika, Minsk, 1966.

2. L. I. Gutenmakher, Electrical Simulation (Electro-Integrator) [in Russian], Izd. AN SSSR, 1943.

3. N. V. Antonishin, L. E. Simchenko, and G. A. Surkov, IFZh [Journal of Engineering Physics], 11, no. 4, 1966.

4. L. I. Rubinshtein, Izv. AN SSSR, ser. geogr. i geofiz., 12, no. 1, 1948.

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